

**FROM THE PÓLYA-SZEGÖ SYMMETRIZATION  
INEQUALITY FOR DIRICHLET INTEGRALS TO  
COMPARISON THEOREMS FOR P.D.E.'S ON MANIFOLDS\***  
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*Disclaimer.* These are only lecture notes.

## 1. THE MANIFOLD CASES UNDER CONSIDERATION

Let  $\mathbb{M}_k^m$  be the simply connected constant curvature space form of dimension  $m$ .

- $\mathbb{M}_0^m$  is  $\mathbb{R}^m$  with euclidean metric
- $\mathbb{M}_k^m$  for  $k > 0$  is an  $m$ -sphere of radius  $k^{-1/2}$
- $\mathbb{M}_k^m$  for  $k < 0$  is  $m$  dimensional hyperbolic space modelled on the  $m$ -ball of radius  $(-k)^{-1/2}$ .

Other definitions:

- all manifolds are Riemannian
- $M$  is an  $m$ -dimensional manifold,  $m \geq 1$
- fix  $k \in \mathbb{R}$
- for  $\Omega \subseteq M$ , let  $\Omega^\#$  be the geodesic ball with the same volume as  $\Omega$  centred about the origin  $O$  in  $\mathbb{M}_k^m$
- fix  $B \subset M$ , with infinitely differentiable boundary and homeomorphic to  $\mathbb{R}^m$
- assume that for any open  $\Omega \subseteq B$  with rectifiable boundary we have the isoperimetric inequality

$$V_{m-1}(\partial\Omega) \geq V_{m-1}(\partial(\Omega^\#)).$$

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This isoperimetric inequality was conjectured by Aubin (1976) if all the sectional curvatures of  $M$  are bounded above by  $k$ . It is true if:

- $M = \mathbb{M}_{k'}^m$  for  $k' \leq k$
- $m = 2$  (Weil's first paper, 1926, etc.)
- $k \leq 0$  and  $m = 3$  (Kleiner, 1992)
- $k = 0$  and  $m = 4$  (Croke, 1984).

Our results assume the inequality and only have substance where the inequality is true.

For a real  $f$  on a set  $X$ , let  $f_t = \{x : f(x) > t\}$ . Given a map  $(\cdot)^\#$  of measurable subsets of  $X$  into measurable subsets of  $Y$ , for  $y \in Y$  let

$$f^\#(y) = \inf\{t : y \in (f_t)^\#\}.$$

If  $(\cdot)^\#$  is measure preserving then  $f^\#$  and  $f$  are equimeasurable. If it is subset-preserving, then the Hardy-Littlewood inequality holds for real  $f$  and  $g$ :

$$\int_X fg \leq \int_Y f^\# g^\#.$$

Let  $f$  be non-negative and smooth on  $M$ , vanishing on  $\partial B$ . Then, it follows from the isoperimetric inequality and coarea formula that

$$\int_B |\nabla f|^2 dV_m \geq \int_{B^\#} |\nabla f^\#|^2 dV_m.$$

If  $M = B = \mathbb{M}_k^m = \mathbb{R}^m$ , this is the Pólya-Szegő Dirichlet-integral symmetrization inequality. Note that  $\int_B |\nabla f|^2 = -\int_B f \Delta f$ .

Let  $N$  be a manifold of dimension  $n \geq 0$ . Given  $\Omega \subseteq M \times N$ , let

$$\Omega^\# = \bigcup_{y \in N} (\Omega(y))^\# \times \{y\},$$

where  $\Omega(y) = \{x : (x, y) \in \Omega\}$ . The set  $\Omega^\# \subseteq \mathbb{M}_k^m \times N$  is a **generalized Steiner symmetrization**. It is precisely Steiner symmetrization if  $M = \mathbb{R}^m = \mathbb{M}_k^m$  and  $N = \mathbb{R}^n$ . The operation  $(\cdot)^\#$  induces a rearrangement on functions as above.

## 2. RESULTS ON MANIFOLDS

**Theorem.** *Let  $\Omega \subseteq B \times N$  have compact closure and nice boundary. Let  $u$  and  $v$  be  $C^2$  and non-negative on  $\Omega$  and  $\Omega^\#$  respectively, vanishing outside their respective domains, and solving*

$$-\Delta u = \phi(u) + \psi u + \lambda$$

and

$$-\Delta v = \phi(v) + \psi^\# u + \lambda^\#,$$

where  $\phi$  is continuous decreasing on  $[0, \infty)$  and  $\lambda$  and  $\psi$  continuous on  $\bar{B} \times N$ . Let  $\Phi$  be convex increasing on  $[0, \infty)$ . Then, for all  $y \in B$  we have

$$\int \Phi(u(x, y)) dV_m(x) \leq \int \Phi(v(x, y)) dV_m(x),$$

and moreover  $v = v^\#$ .

In particular,

$$\max_x u(x, y) \leq v(O, y),$$

where  $O$  is the origin in  $\mathbb{M}_k^m$ .

A similar parabolic theorem can be proved, with the added condition that  $v$  satisfies the symmetrization of the initial condition for  $u$ .

### 3. SOME IDEAS FOR PROOFS

Given  $u$  on  $B \times N$ , define on  $B^\# \times N$ :

$$u^I(x, y) = \int_{\mathbb{B}(d(x, O))} u^\#(x', y) dV_m(x')$$

(Baernstein  $*$ -function), where  $\mathbb{B}(r)$  is the ball around  $O \in \mathbb{M}_k^m$  with geodesic radius  $r$ . Given  $v$  on  $B^\# \times N$ , define on  $B^\# \times N$ :

$$Jv(x, y) = \int_{\mathbb{B}(d(x, O))} v(x', y) dV_m(x').$$

Thus,  $u^I = J(u^\#)$ . The conclusion of our theorem is equivalent to

$$u^I \leq Jv.$$

The proof of the theorem hinges on:

**Proposition.** *Let  $u$  be as in the Theorem. Then,*

$$\int_{\Omega^\#} \vartheta \cdot (-\Delta u^\#) \leq \int_{\Omega^\#} \vartheta \cdot (\phi(u^\#) + \psi^\# u^\# + \lambda^\#),$$

in the distributional sense for every smooth (say  $C^2$ ) function  $\vartheta$  on  $\Omega^\#$  vanishing on the boundary and satisfying  $\vartheta = \vartheta^\#$ .

Assume the Proposition. Suppose we are in the case  $\phi \equiv \psi \equiv 0$ . (The proof extends to the general case by a clever method of Weitsman as in the case of Steiner symmetrization on  $\mathbb{R}^{n+m}$ .) Let  $\vartheta$  be as in the Proposition. Then,

$$\int_{\Omega^\#} \vartheta \cdot (-\Delta v) = \int_{\Omega^\#} \vartheta \cdot \lambda^\#.$$

By the Proposition if  $g = u^\# - v$  then,

$$\int_{\Omega^\#} \vartheta \cdot (-\Delta g) \leq 0.$$

Let  $\alpha$  be a continuous positive function on  $\Omega^\#$ , vanishing on the boundary, and satisfying  $\Delta \alpha = -1$  in  $\Omega^\#$ . Let  $g_\varepsilon = g - \varepsilon \alpha$ . We have

$$\int_{\Omega^\#} \vartheta \cdot \Delta g_\varepsilon \geq \int_{\Omega^\#} \varepsilon \vartheta.$$

We shall show that it follows that  $g_\varepsilon^I \leq 0$  on  $\Omega^\#$ . The Theorem follows from this in the limit as  $\varepsilon \rightarrow 0$ .

Let  $\mathcal{M}$  be the set of measures  $\mu$  on  $\overline{\Omega^\#}$  such that  $\mu(A^\#) \geq \mu(A)$  for all Borel  $A$  and  $\mu(\overline{\Omega^\#}) \leq 1$ . Define

$$G(\mu) = \int g_\varepsilon d\mu.$$

We shall prove that  $G \leq 0$ . This will immediately imply that  $g_\varepsilon^I(x, y) \leq 0$  since  $g_\varepsilon^I(x, y) = |B(d(y, O))|G(\mu_{x,y})$  for an appropriate probability measure  $\mu_{x,y}$ . To prove  $G \leq 0$ , let  $\mu$  be the measure at which  $G$  attains a maximum, and assume that this maximum is strictly positive. Then  $\mu$  has total mass 1. Taking slices carefully we may prove that there is an extremal  $\mu$  which has support contained in  $M \times \{y\}$  for some  $y$  and which is in be proportional to the measure  $V_m$  on  $M$  lifted to  $M \times \{y\}$  and restricted to some set  $B(r) \times \{y\}$ . More precisely, for a continuous  $f$

$$\int f d\mu = \frac{1}{V_m(\mathbb{B}(r))} \int_{V_m(\mathbb{B}(r))} f(x, y) dV_m(x).$$

I claim that the support of such  $\mu$  cannot be contained inside  $\Omega^\#$ . For, if it is then for  $t > 0$  define a measure  $\mu_t$  with density

$$\rho_t(z) = \int_{\Omega^\#} K_t(z, w) d\mu(w),$$

on  $\Omega^\#$ , where  $K_t$  is the heat kernel on  $B^\#$  (Dirichlet boundary conditions). The measure  $\mu_t$  will have total measure at most 1. Let  $\mu_0 = \mu$ . If the support of  $\mu$  is contained in  $\Omega^\#$ , then

$$\left. \frac{d}{dt} \right|_{t=0+} \int_{\Omega^\#} g_\varepsilon d\mu_t = \int_{\Omega^\#} \Delta g_\varepsilon d\mu \geq \varepsilon.$$

Of course this equation does not really make sense since  $\Delta g_\varepsilon$  is only defined distributionally, but we can make it make enough sense by approximating  $\mu$  with measures which have sufficiently smooth density. The last inequality “follows” from the fact that  $\int_{\Omega^\#} \vartheta \Delta g_\varepsilon \geq \varepsilon$  if  $\vartheta$  is positive,  $C^2$ , has  $\vartheta^\# = \vartheta$  and mean 1. Since  $\mu_t \in \mathcal{M}$  as it has desired symmetry because of the symmetry of  $K_t$ , it follows that  $\mu_0$  cannot be the extremal measure.

Now, suppose the extremal measure is proportional to a lifting of the measure on  $M$  to  $\Omega^\#(y) \times \{y\}$  for some  $y$ . Let  $r_0$  be such that  $\Omega^\#(y) = \mathbb{B}(r_0)$ . Assume that  $r_0 > 0$ . (The case  $r_0 = 0$  is easy as  $g_\varepsilon(0, y) = 0$ .) Define  $\nu_r$  to be the measure on  $B(r)$  lifted from the measure on  $M$  so that

$$\int f d\nu_r = \int_{\mathbb{B}(r)} f(x, y) d\nu_r(x).$$

Then,

$$\left. \frac{d}{dr} \right|_{r=r_0} \int g_\varepsilon d\nu_r = \left. \frac{dV_m(\mathbb{B}(r))}{dr} \right|_{r=r_0} \cdot g_\varepsilon(x, y) = 0,$$

where  $x$  was such that  $d(x, O) = r_0$ . Now then, let  $\mu_r$  be  $\nu_r$  normalized to have total mass 1, i.e., let  $\mu_r = (V_m(\mathbb{B}(r)))^{-1} \nu_r$ . It follows that the derivative of

$$G(\mu_r) = \frac{1}{V_m(\mathbb{B}(r))} \int g_\varepsilon d\nu_r$$

with respect to  $r$  from the left at  $r_0$  equals  $G(\mu_{r_0})$  times the derivative with respect to  $r$  of  $V_m(\mathbb{B}(r))^{-1}$ , which derivative is strictly negative. Thus, if  $G(\mu_{r_0}) > 0$  then the derivative of  $G(\mu_r)$  is strictly negative at  $r_0$ , and it follows that  $G(\mu_r) < G(\mu_{r'})$  for some  $r' < r$ . Since  $\mu_{r'} \in \mathcal{M}$  for  $r' \leq r$ , this is a contradiction. Hence  $G(\mu_{r_0}) \leq 0$ , and the proof is complete.

To prove the Proposition, we first need the parabolic case of the Theorem with  $\phi = \lambda = 0$ . This uses a slight extension of the Polyá-Szegö inequality, and in effect has already been done by Bérard and Gallot (1980)<sup>1</sup> This case can be rewritten as:

$$\int_{B^2} f(x) K_t^B(x, y) g(y) dV_{2m}(x, y) \leq \int_{(B^\#)^2} f^\#(x) K_t^{B^\#}(x, y) g^\#(y) dV_{2m}(x, y),$$

for  $f, g \geq 0$  on  $B$ , where  $K_t^B$  is the heat kernel vanishing on the boundary of  $B$ . This inequality is similar to the Riesz-Sobolev inequality. Now,  $K_t^{B \times N}((x_1, x_2), (y_1, y_2)) = K_t^B(x_1, y_1) K_t^N(x_2, y_2)$ . This and Fubini's theorem implies that

$$\begin{aligned} \int_{(B \times N)^2} f(x) K_t^{B \times N}(x, y) g(y) dV_{2(m+n)}(x, y) \\ \leq \int_{(B^\# \times N)^2} f^\#(x) K_t^{B^\# \times N}(x, y) g^\#(y) dV_{2(m+n)}(x, y). \end{aligned}$$

How can we use this to prove our Proposition? Well, let  $\vartheta$  and  $u$  be as in it. Let  $\tilde{\vartheta}$  be a function on  $B \times N$  such that:

- $\tilde{\vartheta}^\# = \vartheta$
- $\tilde{\vartheta}$  is similarly ordered to  $u$  (i.e.,  $\tilde{\vartheta}(x) \leq \tilde{\vartheta}(y)$  iff  $u(x) \leq u(y)$ ; this is equivalent to requiring that  $\int_{\Omega^\#} \tilde{\vartheta}^\# \cdot u^\# = \int_{\Omega} \tilde{\vartheta} \cdot u$ .)

We have

$$\begin{aligned} \int_{\Omega^\#} \vartheta \cdot (-\Delta u^\#) dV_{m+n} \\ = - \lim_{t \rightarrow 0+} \frac{1}{t} \int_{(\Omega^\#)^2} [\vartheta(x) K_t^{B^\# \times N}(x, y) u^\#(y) - \vartheta(x) \delta(x, y) u^\#(y)] dV_{2(m+n)}(x, y) \\ \leq - \lim_{t \rightarrow 0+} \frac{1}{t} \int_{\Omega^2} [\tilde{\vartheta}(x) K_t^{B \times N}(x, y) u(y) - \tilde{\vartheta}(x) \delta(x, y) u(y)] dV_{2(m+n)}(x, y) \\ = \int_{\Omega} \tilde{\vartheta} \cdot (-\Delta u) dV_{m+n} \\ = \int_{\Omega} \tilde{\vartheta} \cdot (\phi(u) + \psi u + \lambda) \\ \leq \int_{\Omega^\#} \vartheta \cdot (\phi(u^\#) + \psi^\# u^\# + \lambda^\#). \end{aligned}$$

#### 4. DISCRETE CASES

The methods used can also give discrete symmetrization theorems. Here,  $M$  and  $N$  are two discrete sets, and a laplacian is defined on  $M \times N$ . Starting with a convolution-rearrangement inequality on  $M$  like the one for the heat kernel in

<sup>1</sup>Not to be confused with [1987]. Cf. Gallot (1988, Th. 5.4(iii)).

the manifold case, one can duplicate most if not all of the above work for a symmetrization on the discrete product set  $M \times N$ , with difference equations instead of p.d.e.'s.

An appropriate convolution-rearrangement inequality is known if  $M$  is:

- the discrete line  $\mathbb{Z}$ , where we reorder functions so that

$$f^\#(0) \geq f^\#(1) \geq f^\#(-1) \geq \dots$$

(Hardy and Littlewood)

- the discrete circle  $\mathbb{Z}_m$ , where we reorder functions so that

$$f^\#(0) \geq f^\#(1) \geq f^\#(-1) \geq \dots$$

(effectively due to J. R. Quine for the standard random walk; extended by the author to more general walks; the limiting case as  $m \rightarrow \infty$  is of course  $\mathbb{Z}$ )

- the  $m$ -regular tree  $T_m$ , with a spiral like reordering (this is due to the author); this inequality also implies a Faber-Krahn inequality for subsets of the  $m$ -regular tree
- the edge graph of an octahedron.

All four convolution-rearrangement inequalities can be proved by a discrete analogue of a method of Baernstein and Taylor (1976), generalized (still in the continuous case) by Beckner (1993).

However, in the discrete case the method cannot handle many situations. For instance, even the analogue of the Dirichlet integral inequality fails on the graphs  $\mathbb{Z}_2^3$  (cube) and  $\mathbb{Z}_3^2$  (a euclidean plane based on a finite field).<sup>2</sup>

## 5. REMARKS ADDED IN 1997

It is worth noting that while the above symmetrization methods symmetrize a manifold  $M$  by using a manifold of revolution modelled on the isoperimetric relations in  $M$  (see Gallot, 1988) instead of  $\mathbb{M}_k^n$ .

The question of the general results that these kinds of methods can give on manifolds is still open and the reader is invited to explore this further.<sup>3</sup> The present notes merely outline the method. Further research on the manifold cases could probably make use of the analogous but fully worked-out version of the method in the discrete case (Pruss, 1997b; see especially Technical Remark 3.1 for connections to manifolds).

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<sup>2</sup>Note added in 1997: See Pruss, 1997a, 1997b, and 1997c for details of the discrete work.

<sup>3</sup>The reader should be able to find functions that do this, if they are willing to study the isoperimetric

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